

Some New Thoughts on Maximal Functions and Poisson Integrals

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Abstract: *We study Wiener-type covering lemmas, Hardy-Littlewood-type maximal functions, and convergence theorems on metric spacs. Later we specialize down to a result for the Poisson integral. We show that, in a suitably general setting, these three phenomena are essentially logically equivalent. Along the way we discuss some useful estimates for the Poisson kernel.*

0 Introduction

A standard paradigm in harmonic analysis—on Euclidean space, or on a space of homogeneous type—is that if one can prove a covering lemma of Wiener type then one can prove a weak-type estimate for a suitable maximal function. And this, in turn, will imply pointwise convergence results for certain convolution operators. The celebrated “limits of sequences of operators”

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theorem of Stein [STE1] fleshes out this picture by showing that, under suitable hypotheses, a pointwise convergence result implies a maximal function estimate. One of our goals in this paper is to complete this logical scenario. In particular, we wish to show that if one has pointwise convergence of integral operators, then one can derive a covering lemma. Thus, in effect, the three key ideas being discussed here are logically equivalent.

We begin our investigations with a few words about the size of the Poisson kernel.

1 Estimates on the Poisson Kernel

Of course the Poisson kernel on classical domains is well known. For example,

- The Poisson kernel of the disc $D \subseteq \mathbb{R}^2$ is

$$P_D(z, \zeta) = \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - \zeta|^2},$$

where $z \in D$ and $\zeta \in \partial D$.

- The Poisson kernel for the upper halfplane

$$U^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$$

is given by

$$P_{U^2}(x, t) = \frac{1}{\pi} \cdot \frac{x_2}{(x_1 - t)^2 + x_2^2},$$

where $x = (x_1, x_2) \in U^2$ and $t \in \mathbb{R} = \partial U^2$.

- The Poisson kernel for the unit ball $B \subseteq \mathbb{R}^N$ is given by

$$P_B(x, t) = \frac{\Gamma(N/2)}{2\pi^{N/2}} \cdot \frac{1 - |x|^2}{|x - t|^N},$$

where $x = (x_1, x_2, \dots, x_N) \in B$ and $t = (t_1, \dots, t_N) \in \partial B$. Here Γ is the classical gamma function.

- The Poisson kernel for the upper halfspace $U^{N+1} \equiv \{x = (x_1, \dots, x_{N+1}) \in \mathbb{R}^{N+1} : x_{N+1} > 0\}$ (with $x = (x_1, \dots, x_{N+1}) \equiv (x', x_{N+1})$) is given by

$$P_{U^{N+1}}(x, t) = c_N \frac{x_{N+1}}{([x' - t]^2 + x_{N+1}^2)^{[N+1]/2}}$$

where $x = (x_1, x_2, \dots, x_{N+1}) \in U^{N+1}$, $t = (t_1, t_2, \dots, t_N) \in \mathbb{R}^N = \partial U^{N+1}$, and

$$c_N = \frac{\Gamma([N+1]/2)}{\pi^{[N+1]/2}}.$$

We say that $\Omega \subseteq \mathbb{R}^N$ is a *domain* if it is a connected, open set. Of course there is no hope of producing an actual formula for the Poisson kernel of an arbitrary domain in \mathbb{R}^N . But there is definite interest in obtaining an *asymptotic* formula for the Poisson kernel on a general domain. The standard asymptotic (see [STE3] or [KRA1]) is

$$P_\Omega(x, t) \approx \frac{\delta(x)}{|x - t|^N} \quad (*)$$

for $x \in \Omega$ and $t \in \partial\Omega$. Here $\delta(x) \equiv \delta_\Omega(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$. This estimate, together with analogous estimates for the derivatives of P_Ω , suffices for most applications. Stein states this result in [STE3], but does not prove it. The reference that he gives is also incomplete in this matter. Perhaps the first complete proof to appear in print can be found in [KRA1]. A more recent, and more efficient, proof appears in [KRA2]. In fact a very natural way to approach the matter is to fix a point x in Ω near the boundary and to consider a small, topologically trivial subdomain $\Omega' \subseteq \Omega$ which contains x and which shares a piece W of boundary with Ω . It follows from basic estimates on the Green's function (see, for instance, [APF]) that the Poisson kernel of Ω for this fixed point x and for $t \in W$ is comparable to the Poisson kernel of Ω' for that same x and t (the constants of comparison, of course, depend on Ω and Ω'). Now Ω' may be mapped diffeomorphically to the unit ball by a mapping Φ . And the Poisson kernel P of the unit ball may be pulled back to Ω' by means of pseudodifferential operator theory. One obtains immediately that

$$P_{\Omega'}(x, y) \approx \frac{\delta_{\Omega'}(x)}{|x - y|^N} + \mathcal{E}(x, y),$$

where $\mathcal{E}(x, y)$ is an error term of lower order. The result follows.

We will make good use of the approximation $(*)$ in the sequel.

2 Principal Results

We begin by enunciating the results that are known. We do so in the setting of an arbitrary metric space (*not* a space of homogeneous type).

We first need some definitions. In what follows, let (X, ρ) be a metric space. The balls in this metric space will be denoted by

$$B(x, r) \equiv \{t \in X : \rho(x, t) < r\}.$$

We take it that the metric space is equipped with a Borel regular measure (i.e., a Radon measure) μ , and that the measure of each ball is positive and finite. But we do *not* assume that (X, ρ, μ) is a space of homogeneous type in the sense of [COW]. Sometimes, for convenience, we will denote the μ -measure of a ball $B(x, r)$ by $|B(x, r)|$ or, more generally, the μ -measure of a measurable set S by $|S|$.

Definition 1 We say that the balls $B(x, r)$ satisfy a *Wiener-type covering lemma* if, whenever $K \subseteq X$ is compact and $\{B_\alpha\}_{\alpha \in A}$ is a covering of K by balls, then there is a subcollection $\{B_{\alpha_j}\}_{j=1}^K$ such that

- (a) The $\{B_{\alpha_j}\}$ are pairwise disjoint.
- (b) The dilated balls $\{3B_{\alpha_j}\}$, where $3B(x, r) \equiv B(x, 3r)$, cover K .

Definition 2 A real-valued function f on an open set $U \subseteq X$ is said to be of *weak type* α , $\alpha > 0$, if there is a $C > 0$ such that, for any $\lambda > 0$,

$$\mu\{x \in U : |f(x)| > \lambda\} \leq \frac{C}{\lambda^\alpha}.$$

It is easy to see that any L^α function is in fact weak type α .

Definition 3 A linear operator defined on $L^p(U)$, for some open set $U \subseteq X$, and taking values in the real functions on some set $V \subseteq X$, is said to be of *weak type* (p, p) if there is a constant $C > 0$ so that, for any $\lambda > 0$,

$$\mu\{x \in V : |Tf(x)| > \lambda\} \leq C \cdot \left[\frac{\|f\|_{L^p(U)}}{\lambda} \right]^p. \quad (\star)$$

We say that the linear operator is of *restricted weak type* (p, p) if it satisfies the “weak type (p, p) ” condition (\star) when restricted to act on f the characteristic function of a set.

Definition 4 For a real-valued, locally integrable function f defined on an open $U \subseteq X$, and $x \in U$, define

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(t)| dt.$$

We call M the *Hardy-Littlewood maximal operator* or *Hardy-Littlewood maximal function*.

Definition 5 Let $\mathcal{B} = \{B_\alpha\}_{\alpha \in A}$ be a covering of some set E by balls. We call $\mathcal{C} = \{C_\beta\}_{\beta \in B}$ a *refinement* of the covering \mathcal{B} if each element C_β of \mathcal{C} is a subset of some $B_\alpha \in \mathcal{B}$ and if $\cup_\beta C_\beta \supset E$.

Definition 6 We say that the *Lebesgue differentiation theorem* holds in $L^p(X)$ if, for any $f \in L^p(X)$, and for almost every $x \in X$, it holds that

$$\lim_{r \rightarrow 0^+} \frac{1}{|B(x, r)|} \int_{B(x, r)} f(t) dt$$

exists and equals f .

Now we have

Proposition 1 Assume that there is a constant $C > 0$ so that

$$\mu(B(x, 3r)) \leq C \cdot \mu(B(x, r))$$

for any $x \in X$ and any $r > 0$. Then the fact that metric balls $B(x, r)$ satisfy a Wiener-type covering lemma implies that the Hardy-Littlewood maximal operator is weak type $(1, 1)$.

The fact that the Hardy-Littlewood maximal operator is weak type $(1, 1)$ implies that the Lebesgue differentiation theorem holds in L^1 .

Under suitable additional hypotheses, the fact that the Lebesgue differentiation theorem holds for functions in L^1 implies that the Hardy-Littlewood maximal operator is weak type $(1, 1)$.

These statements are all classical. The third one is—in the classical setting of the circle group, for instance—the content of Stein’s limits of sequences of operators theorem [STE1]. In our more general setting of metric spaces we would want to apply instead Sawyer’s generalization of this result [SAW] or

Nikisin's generalization of this result [GIL]; this would require the averaging operators to satisfy some ergodic or mixing condition.

Our intention now is to round out the logic in this last proposition. We shall show that the fact that the Lebesgue differentiation theorem holding for functions in L^1 implies that metric balls $B(x, r)$ satisfy a covering lemma.

We have intentionally omitted the phrase “Wiener-type” from the enunciation of this last result because in fact we will need to formulate a new covering lemma in order to make the logic work. This new covering lemma is of intrinsic interest, and we shall spend some time putting it into context.

3 The New Covering Lemma

In order for the logic to mesh properly, we need to formulate a new type of covering lemma. Then we need to show that it does the same job as the old (Wiener-type) covering lemma. The new lemma is this (we work still on the metric space (X, ρ) , which is *not* necessarily a space of homogeneous type):

Lemma 2 *The Lebesgue differentiation theorem implies the following covering lemma: Let $\{B_\alpha\}_{\alpha \in A}$ be a covering of a compact set K by open metric balls. Then there is a refinement $\{C_j\}$ of the covering so that*

(a) *The balls C_j are pairwise disjoint.*

(b) *Each ball C_j has the property that*

$$\frac{|C_j \cap K|}{|C_j|} > \frac{1}{2};$$

(c) *We have the inequality*

$$\sum_j |C_j| > \frac{1}{2}|K|;$$

Proof: Fix a compact set $K \subseteq X$. For almost every $x \in K$, the Lebesgue differentiation theorem tells us that there is an $r_x > 0$ so that, if $0 < s \leq r_x$ then

$$|B(x, s) \cap K| > \frac{1}{2}|B(x, s)|.$$

By Borel regularity, let U be an open set that contains K and so that $|U \setminus K| < (1/10) \cdot |K|$. We may assume that each r_x is so small that $B(x, r_x) \subseteq U$ for each $x \in K$.

Now if $x \in U \setminus K$ then take $r_x = \min(\text{dist}(x, K), \text{dist}(x, {}^cU)$. We may apply the (proof of) the Vitali covering theorem now to extract a subcollection $\{B(x_j, r_{x_j})\}$ so that the $B(x_j, r_{x_j})$ are pairwise disjoint and so that $\sum_j |B(x_j, r_{x_j})| > (1/2)K$. \square

Proposition 3 *The Lebesgue differentiation theorem implies that the Hardy-Littlewood maximal operator is restricted weak type $(1, 1)$.*

Proof: Fix a measurable set E with $|E| < \infty$. Now let $0 < \lambda < 1$ and set

$$S_\lambda = \{x \in X : M\chi_E > \lambda\}.$$

Let $K \subseteq S_\lambda$ be compact and satisfy $|K| > |S_\lambda|/2$. For each $x \in K$ there is a ball B_x such that

$$\frac{1}{|B_x|} \int_{B_x} \chi_E(t) dt > \lambda.$$

Now, as in the proof of Lemma 2, choose a refinement $\{B_{x_j}\}$ so that, for each j ,

$$\frac{|B_{x_j} \cap K|}{|B_{x_j}|} > \frac{1}{2}$$

and so that the $\{B_{x_j}\}$ are pairwise disjoint. Finally we may assume that $\sum_j |B_{x_j}| > |K|/2$, again by arguing as in the Vitali covering lemma.

Then

$$\begin{aligned} |S_\lambda| &\leq 2|K| \\ &\leq 4 \sum_j |B_{x_j}| \\ &\leq \frac{4}{\lambda} \cdot \sum_j |B_{x_j} \cap E| \\ &\leq \frac{4|E|}{\lambda}. \end{aligned}$$

That is the estimate that we want. So M is restricted weak type $(1, 1)$. \square

Now the sum of what we have established thus far is this:

$$\begin{aligned}
& \text{(covering lemma)} \implies \text{(maximal function estimate)} \\
& \implies \text{(differentiation theorem)} \implies \text{(covering lemma)} \\
& \implies \text{(maximal function estimate)}
\end{aligned}$$

So we have closed the logical circle described at the beginning of this paper.

4 More General Results

In fact the Lebesgue differentiation theorem is merely a paradigm for the type of result that is true in general. We first introduce some notation and terminology.

If $\Omega \subseteq \mathbb{R}^N$ is a smoothly bounded domain, then we let $d\sigma$ denote $(N - 1)$ -dimensional Hausdorff measure on $\partial\Omega$. For $x \in \Omega$, let $\delta(x)$ denote the distance of x to $\partial\Omega$. If $y \in \partial\Omega$ and $r > 0$ then let $\beta(y, r) = \{t \in \partial\Omega : |t - y| < r\}$. If $y \in \partial\Omega$ and $\alpha > 0$, then let

$$\Gamma_\alpha(y) = \{x \in \Omega : |x - y| < \alpha \cdot \delta(x)\}.$$

Theorem 4 *Let Ω be a smoothly bounded domain in \mathbb{R}^N . Let $P(x, t)$ be the Poisson kernel for Ω , with $x \in \Omega$ and $t \in \partial\Omega$. If $f \in L^1(\partial\Omega, d\sigma)$, then set*

$$u(x) = \int_{\partial\Omega} P(x, t) f(t) d\sigma(t).$$

Fix $\alpha > 0$. The σ -almost everywhere existence of the boundary limits

$$\lim_{\Gamma_\alpha(y) \ni x \rightarrow y} u(x) = f(y)$$

is logically equivalent to the restricted weak-type $(1, 1)$ property of the maximal operator

$$\mathcal{M}f(y) \equiv \sup_{r>0} \frac{1}{\sigma(\beta(y, r))} \int_{\beta(y, r)} |f(t)| d\sigma(t).$$

Proof: That the restricted weak-type $(1, 1)$ property of the \mathcal{M} implies the almost everywhere boundary limit result is standard.

The proof of the reverse direction follows the lines in the last section, using of course the asymptotic estimate $(*)$ for P . In particular, it is straightforward to see that

$$\lim_{r \rightarrow 0^+} \frac{1}{\sigma(\beta(y, r))} \int_{\beta(y, r)} f(t) d\sigma(t) = f(y)$$

if and only if

$$\lim_{\Gamma_\alpha(y) \ni x \rightarrow y} \int_{\partial\Omega} P(x, t) f(t) d\sigma(t) = f(x).$$

Indeed, it is standard to compare the Poisson kernel to a weighted sum of characteristic functions of balls (the inequality $(*)$ facilitates this comparison), and that gives the result. That completes the proof. \square

5 Concluding Remarks

The results of this paper round out the picture of the equivalence of various key results in basic harmonic analysis. They will come as no surprise to the experts, but it is useful to have the ideas recorded in one place, and to know that all these key ideas are logically equivalent.

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